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Entanglement measurement with discrete multiple-coin quantum walks

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Abstract

Within a special multi-coin quantum walk scheme, we analyse the effect of the entanglement of the initial coin state. For states with a special entanglement structure, it is shown that this entanglement can be measured with the mean value of the walk, which depends on the i-concurrence of the initial coin state. Further, the entanglement evolution is investigated and it is shown that the symmetry of the probability distribution is reflected by the symmetry of the entanglement distribution.

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1. Introduction

Quantum walks are the quantum counterparts of classical random walks. Since they were introduced in [1], they were investigated to construct the same powerful algorithms for future quantum computers as in the classical case. An overview can be found in [2]. Here we are interested in an entanglement measure and will study a special constructed discrete quantum-walk scheme with more than one coin. We will connect the entanglement of the initial coin state with the outcoming mean value in position space. An equation will be given that establishes the mean value as an entanglement measure for the initial coin state. Further, we will investigate the evolution of the entanglement in coin space.

1.1. Quantum-walk scheme

We use a simplified quantum-walk scheme with M coins, which is in its base similar to the model used by Brun *et al* [3]. There are two main differences in our model. On the one hand, we are using a different coin-tossing operator and on the other we apply the operator to all the qubits one after the other. With this, it is possible to construct an entanglement measurement scheme.

The Hilbert space consists of two parts: a $(M \times 2)$ -dimensional coin space $H_C^{\otimes M}$ describing the M 2-level coins and an n -dimensional position space H_P describing the one-dimensional lattice with n sites. The quantum-walk scheme consists only of applying repeatedly a special transformation operator E to an initial state $|\Psi(t = 0)\rangle \equiv |\Psi_0\rangle$

$$|\Psi(t)\rangle = E^t |\Psi_0\rangle. \tag{1}$$

The wavefunction $|\Psi_0\rangle$ consists of two parts: a part describing the coin state and the other giving the probability of the walker to be at a certain lattice site, $|\Psi_0\rangle = |\psi_C\rangle \otimes |\psi_P\rangle$. The transformation operator E has two parts: one for the coin tossing and the other for the shifting of the walker on the lattice. As an example we show here an operator with $M = 3$ coins, which executes the operation depending on the state of qubit 2:

$$E = \left(\underbrace{I_C \otimes |0\rangle\langle 0| \otimes I_C}_{\text{qubit 2}} \otimes S^{+1} + \underbrace{I_C \otimes |1\rangle\langle 1| \otimes I_C}_{\text{qubit 2}} \otimes S^{-1} \right) \left(\underbrace{I_C \otimes U \otimes I_C}_{\text{qubit 2}} \otimes I_P \right). \tag{2}$$

We use the unbiased tossing operator

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}, \tag{3}$$

which means that after one step we have the same probability for going one step to the left or one step to the right. The shifting operators move the quantum walker on the one-dimensional lattice, $S^+ = \sum_x |x + 1\rangle\langle x|$ and $S^- = \sum_x |x - 1\rangle\langle x|$, where x marks the position. The $I_{\{P,C\}}$ are unity operators in the position resp. the coin subspace. One can also apply the operation to qubit 1 resp. qubit 3. By introducing more I_C operators in coin space, one can extend the E operator to more coins.

2. Analytical evaluation

2.1. Fourier transformation

We use the methods described by Nayak and Vishwanath [4] and by Brun and co-workers [3] to evaluate our scheme analytically. After applying a discrete Fourier transformation to the quantum-walk scheme, we get the U operator in k -space:

$$U_k = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{ik} & i e^{ik} \\ i e^{-ik} & e^{-ik} \end{pmatrix}. \tag{4}$$

The time evolution can then easily be calculated by the eigenvalue decomposition of the U_k operator. The calculation of the eigenvalues yields $\lambda_{1,2} = \frac{1}{\sqrt{2}}(\cos k \pm i\sqrt{1 + \sin^2 k})$. If we apply a similarity transformation with

$$T = \begin{pmatrix} c_+ & c_- \\ 1 & 1 \end{pmatrix} \tag{5}$$

and $c_{\pm} = e^{ik}(\sin k \pm \sqrt{1 + \sin^2 k})$, we can easily calculate the matrix product of the U_k operator:

$$U_k^t = T \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}^t T^{-1} = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix} \tag{6}$$

with the abbreviations $a = \cos t\theta + i \frac{\sin k}{\sqrt{1 + \sin^2 k}} \sin t\theta$, $b = \frac{ie^{ik}}{\sqrt{1 + \sin^2 k}} \sin t\theta$, $\theta = \arccos \frac{\cos k}{\sqrt{2}}$, with $|a|^2 + |b|^2 = 1$.

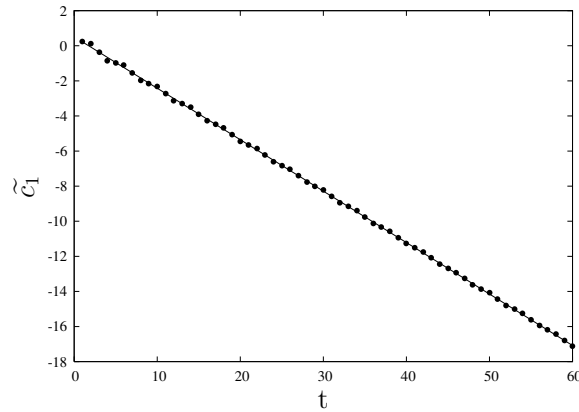


Figure 1. The numerical results of the integral \tilde{c}_1 as a function of time and linear fit with $\tilde{c}_1(t) = a_0 t + a_1$ with $a_0 = -0.2932$ and $a_1 = 0.5116$.

2.2. Calculation of the moments

Brun *et al* [3] derived an equation for the calculation of the moments of the distribution in position space

$$\langle x^m \rangle = \frac{i^m}{2\pi} \int_{-\pi}^{\pi} dk \langle \phi_0 | (U_k^\dagger)^t \left[\frac{d^m}{dk^m} U_k^t \right] | \phi_0 \rangle \quad (7)$$

which we will use to calculate the mean value ($m = 1$) and the variance ($m = 2$). $|\phi_0\rangle$ is the initial state in coin space. We start with the simplification of the integrand for the calculation of the first moment:

$$\begin{aligned} (U_k^\dagger)^t \frac{d}{dk} U_k^t &= \begin{pmatrix} a^* & -b \\ b^* & a \end{pmatrix} \begin{pmatrix} a' & b' \\ (-b^*)' & (a^*)' \end{pmatrix} \\ &= \begin{pmatrix} a'a^* + b(b^*)' & a^*b' - (a^*)'b \\ a'b^* - a(b^*)' & a(a^*)' + b'b^* \end{pmatrix} := \begin{pmatrix} c_1 & d_1 \\ -d_1^* & c_1^* \end{pmatrix}. \end{aligned} \quad (8)$$

We find a special property of c_1 since

$$\begin{aligned} c_1 + c_1^* &= a^*a' + bb'^* + a'^*a + b'b^* \\ &= (aa^*)' + (bb^*)' = (|a|^2 + |b|^2)' = 0 \end{aligned} \quad (9)$$

and therefore $\frac{i}{2\pi} \int_{-\pi}^{\pi} dk (c_1 - c_1^*) = 0$. In the following, we will use the abbreviation

$$\tilde{c}_1 := \frac{i}{2\pi} \int_{-\pi}^{\pi} dk c_1. \quad (10)$$

From the symmetry of the integrand, we can further conclude that $\frac{i}{2\pi} \int_{-\pi}^{\pi} dk (d_1 - d_1^*) = 0$. Our results can easily be verified with these two simplifications. In figure 1, we evaluated numerically the integral \tilde{c}_1 as a function of time and made a linear regression of the result.

For the calculation of the variance and of the mean deviation, one needs the second moment. For this we evaluate the integrand of (7) for $m = 2$.

$$(U_k^\dagger)^t \frac{d^2}{dk^2} U_k^t = \begin{pmatrix} a^* & -b \\ b^* & a \end{pmatrix} \begin{pmatrix} a'' & b'' \\ (-b^*)'' & (a^*)'' \end{pmatrix} \quad (11)$$

$$= \begin{pmatrix} a''a^* + b(b^*)'' & a^*b'' - (a^*)''b \\ a''b^* - a(b^*)'' & a(a^*)'' + b''b^* \end{pmatrix} := \begin{pmatrix} c_2 & d_2 \\ -d_2^* & c_2^* \end{pmatrix}. \quad (12)$$

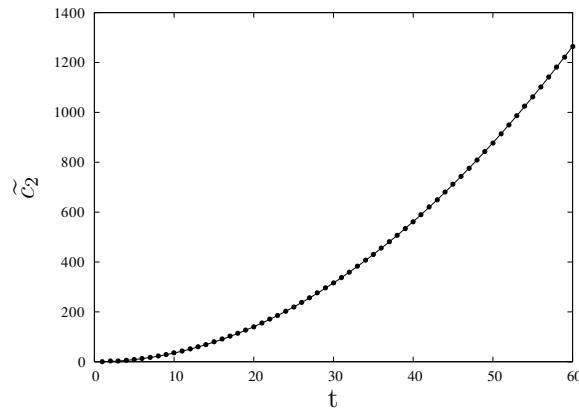


Figure 2. The numerical results of the integral \tilde{c}_2 as a function of time and fit with $\tilde{c}_2(t) = b_0 t^2$ with $b_0 = 0.351\,249$.

We know from numerical evaluation that the integral over the matrix $(U_k^\dagger)^t \frac{d^2}{dk^2} U_k^t$ is diagonal and that the entries have the same value: $-\frac{1}{2\pi} \int_{-\pi}^{\pi} dk (c_2 - c_2^*) = 0$. The integrals over the nondiagonal entries vanish. With these two information, we can conclude that the second moment does not depend on the initial state, but only on the integral

$$\tilde{c}_2 := -\frac{1}{2\pi} \int_{-\pi}^{\pi} dk c_2. \quad (13)$$

This is in accordance with the results obtained by Konno [5, 6]. In figure 2, we evaluated numerically the integral \tilde{c}_2 as a function of time and fitted the result with a quadratic dependence.

We now apply our quantum-walk scheme to five initial coin states, which differ in the entanglement structure. It is shown that for states with a certain entanglement structure, the following equation for the squared mean value is valid:

$$\langle x \rangle_i^2 = \tilde{c}_1^2(t) (1 - IC_i^2) \quad (14)$$

$\langle x \rangle_i$ is the mean value related to the E operator which acts on qubit i . IC_i is the i -concurrence [7] related to the i th qubit of the initial coin state. In addition, the following equation for the variance can be derived:

$$(\langle x^2 \rangle - \langle x \rangle^2)_i = (\tilde{c}_2(t) - \tilde{c}_1^2(t)) + \tilde{c}_1^2(t) IC_i^2. \quad (15)$$

The i -concurrence was introduced by Rungta *et al* [7] to extend the well-known 2-qubit concurrence [8, 9] to larger qubit systems and to measure the entanglement between two subsystems A and B. The i -concurrence can be written as $IC_{A-B} = \sqrt{2[1 - \text{Tr}(\rho_A^2)]}$, with the reduced density matrix $\rho_A = \text{Tr}_B(\rho_{AB})$. We use the notation $IC_{A-B} \equiv IC_A$. In the case of two qubits, the i -concurrence is equivalent to the concurrence, a 2-qubit entanglement measure. In the case of three qubits, it is easy to show that the i -concurrence (reduced to one qubit) can be written as a sum of the 2-qubit concurrences and of the tangle [11], a 3-qubit entanglement measure.

Our main equation (14) connects the mean value of the quantum walk with this i -concurrence for the initial state. We show analytical and numerical examples that equation (14) only holds for so-called pure entangled states, that means states with only one kind of entanglement, e.g. for a tripartite state either 2-qubit entanglement or 3-qubit entanglement. For states with mixed entanglement, the mean value stays zero.

Table 1. The entanglement structure of the example states. 2-qubit entanglement as measured by the concurrence [8, 9], 3-qubit-entanglement for tripartite states as measured by the tangle [11] and 3-qubit resp. 4-qubit entanglement for fourpartite states as described in [10]. With pure entanglement, we mean that the state contains only one kind of entanglement.

State	Equation	2-qubit entanglement	3-qubit entanglement	4-qubit entanglement	Pure entanglement	$\langle x \rangle_i^2 \propto IC_i^2$
$ \gamma\text{GHZ}\rangle$	(16)	No	Yes		Yes	Yes
$ \psi_6\rangle$	(18)	Yes	No		Yes	Yes
$(\psi_7\rangle + \psi_8\rangle)/\sqrt{2}$	(21)	Yes	Yes		No	No
$ \phi_1\rangle$	(23)	Yes	No	No	Yes	Yes
$ \phi_2\rangle$	(25)	No	Yes?	Yes?	No	No

Table 2. The parameter A_0 of the fit of the squared mean values, with $\langle x \rangle_i^2 = A_0(1 - IC_i^2)$, for $t = 50$. $|\phi_1\rangle_1$ resp. $|\phi_1\rangle_2$ means the state $|\phi_1\rangle$ dependent on one parameter, when the other one is constant.

State	IC_1	IC_2	IC_3	IC_4
$ \gamma\text{GHZ}\rangle$	202.634	202.634	202.634	
$ \psi_6\rangle$	202.631	202.631	202.631	
$ \phi_1\rangle_1$	202.639	202.636	202.639	202.634
$ \phi_1\rangle_2$	202.641	202.635	202.641	202.633

3. Application to example states

We will apply our scheme to pure 3- and 4-qubit initial states, which differ in their entanglement structure. This known entanglement structure of these states is described in [10] and summarized in table 1. We start with a parameter-dependent GHZ [12] state

$$|\gamma\text{GHZ}\rangle = \gamma|000\rangle + \sqrt{1 - \gamma^2}|111\rangle \tag{16}$$

which is genuine tripartite entangled for $\gamma \in]0, 1[$. The i-concurrence is the same for all reduced qubits, $IC_{\{1,2,3\}}^2 = 4\gamma^2(1 - \gamma^2)$. The analytical solution for the mean value is found to be

$$\langle x \rangle_{\{1,2,3\}}^2 = \tilde{c}_1^2(2\gamma^2 - 1)^2 = \tilde{c}_1^2(1 - IC_{\{1,2,3\}}^2) \tag{17}$$

and we have proven the result given in (14). In table 2, we give additionally the fit parameter A_0 which comes out of the fit of the simulated mean value with $\langle x \rangle_i^2 = A_0(1 - IC_i^2)$. As one can see the simulation and the analytical value fit very well. The state $|\psi_6\rangle$ is a pure 2-qubit entangled parameter-dependent eigenstate of a certain spin chain and has the following form,

$$|\psi_6\rangle = \kappa_1|001\rangle + \kappa_2|010\rangle + \kappa_1|100\rangle \tag{18}$$

with the norm $2\kappa_1^2 + \kappa_2^2 = 1$ and the abbreviations $\kappa_1 = \frac{\sqrt{\chi}}{2\sqrt{\eta}}$ and $\kappa_2 = -\frac{2}{\sqrt{\chi}\sqrt{\eta}}$ with $\eta := \sqrt{12 + \Delta(\Delta - 4)}$ and $\chi := \eta + \Delta - 2$. The i-concurrence can be calculated in terms of the parameters, $IC_{\{1,3\}}^2 = 1 - \kappa_2^4$ resp. $IC_2^2 = 4(\kappa_2^2 - \kappa_2^4)$. Again it can be shown that the mean values fulfil equation (14):

$$\langle x \rangle_{\{1,3\}}^2 = \tilde{c}_1^2\kappa_2^4 = \tilde{c}_1^2(1 - IC_{\{1,3\}}^2) \tag{19}$$

$$\langle x \rangle_2^2 = \tilde{c}_1^2(1 - 2\kappa_2^2)^2 = \tilde{c}_1^2(1 - IC_2^2). \tag{20}$$

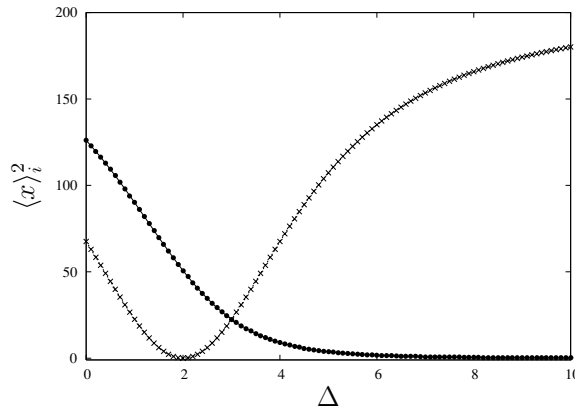


Figure 3. The parameter dependence of the squared mean value for the state $|\psi_6\rangle$ for time $t = 50$. The dots mark qubit 1 resp. 3, the ‘x’ mark qubit 2. The lines are fits with $\langle x \rangle_i^2 = A_0(1 - IC_i^2)$.

In figure 3, we show the parameter dependence of the squared mean value and the result of the fit.

The next state at which we will have a closer look is a superposition of two states:

$$(|\psi_7\rangle + |\psi_8\rangle)/\sqrt{2} = (\kappa_1|001\rangle + \kappa_2|010\rangle + \kappa_1|011\rangle) = \kappa_1|100\rangle + \kappa_2|101\rangle + \kappa_1|110\rangle/\sqrt{2} \quad (21)$$

with abbreviations and norm from above. As we have shown in [10], the entanglement structure of this state shows 2-qubit as well as 3-qubit entanglement. The i-concurrences for this state could easily be calculated, but are not so important in this case, because the mean value is 0, independent from any parameter:

$$\langle x \rangle_{\{1,3\}} = (\tilde{c}_1 + \tilde{c}_1^*) + 2\kappa_1\kappa_2(\tilde{d}_1 - \tilde{d}_1^*) = 0 \quad \langle x \rangle_2 = (\tilde{c}_1 + \tilde{c}_1^*) + 2\kappa_1^2(\tilde{d}_1 - \tilde{d}_1^*) = 0. \quad (22)$$

These calculations are consistent with numerical simulations. Thus it is confirmed that for mixed entanglements the mean value stays at zero.

The two 4-qubit states $|\phi_1\rangle$ and $|\phi_2\rangle$ are also eigenstates of a spin chain [10]. The parameters α_i and β_i depend on two further parameters. The state $|\phi_1\rangle$,

$$|\phi_1\rangle = \alpha_1|1110\rangle + \alpha_2|1011\rangle + \alpha_3|0111\rangle - \alpha_3|1101\rangle \quad (23)$$

with the norm $\alpha_1^2 + \alpha_2^2 + 2\alpha_3^2 = 1$, is only twopartite entangled. For simplicity, we will only regard the i-concurrence reduced on qubits 1 and 3, $IC_{\{1,3\}}^2 = 4(\alpha_3^2 - \alpha_3^4)$. It is again nicely seen that the mean value can be described with our equation (14):

$$\langle x \rangle_{\{1,3\}}^2 = \tilde{c}_1^2(2\alpha_3^2 - 1)^2 = \tilde{c}_1^2(1 - IC_{\{1,3\}}^2). \quad (24)$$

For the two other mean values can also be shown that they are connected to the i-concurrences, via equation (14).

The entanglement structure for the state $|\phi_2\rangle$,

$$|\phi_2\rangle = -\beta_1|0011\rangle + \beta_1|0110\rangle - \beta_1|1001\rangle + \beta_1|1100\rangle - \beta_2|0101\rangle + \beta_2|1010\rangle \quad (25)$$

with the norm $4\beta_1^2 + 2\beta_2^2 = 1$, is a more complex state, since there is no possibility of showing a genuine 3-qubit or 4-qubit entanglement in a fourpartite state. But in [10] we have shown that this state has, besides the 2-qubit entanglement, either 3-qubit or 4-qubit entanglement. The i-concurrences reduced to one qubit are parameter independent, $IC_{\{1,2,3,4\}}^2 = 1$. It can be shown for this state that the mean values are 0,

$$\langle x \rangle_{\{1,2,3,4\}} = (2\beta_1^2 + \beta_2^2)(\tilde{c}_1 + \tilde{c}_1^*) = 0. \quad (26)$$

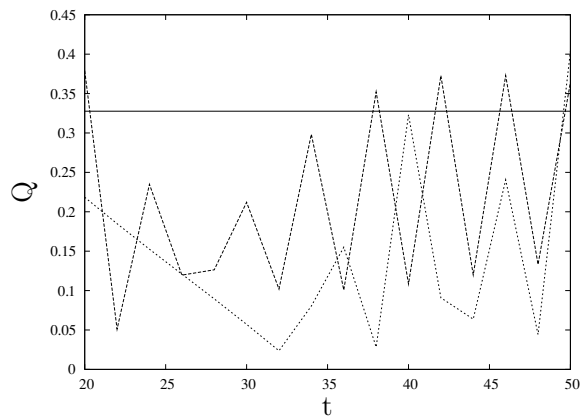


Figure 4. The global entanglement Q as a function of time for the γ GHZ state as initial state, with $\gamma = 0.3$, viewed from different lattice points. The straight line is for $x = 60$, the dashed line for $x = 50$ and the dotted line for $x = 40$. The starting point is at $x = 60$.

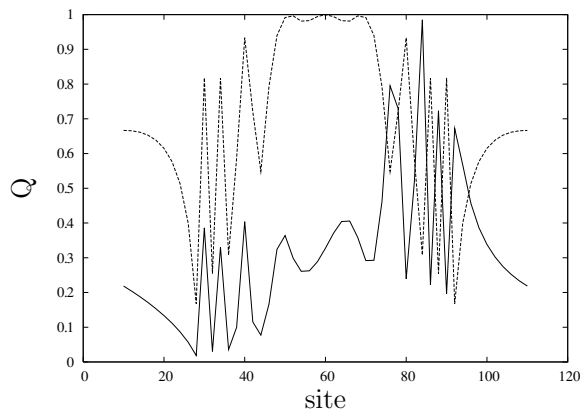


Figure 5. Distribution of the global entanglement Q over the lattice for $t = 50$ for two different γ -parameters for the γ GHZ state as initial state with $\gamma = 1/\sqrt{2}$ (dashed line) and $\gamma = 0.3$ (straight line).

These results are consistent with the 3-qubit problem and therefore we suggest the following general result, although this is not a general proof.

For purely entangled coin-states (only 2-qubit entangled, only 3-qubit entangled, . . .), we expect a square mean value proportional to the corresponding i-concurrence, while for mixed entanglements, the squared mean value is always zero.

One cannot conclude from our results if there exists a smooth cross-over in the square mean value. The 3-qubit state which shows a cross-over in the entanglement structure, $(|\psi_7\rangle + |\psi_8\rangle)/\sqrt{2}$, leads to a zero mean value, independent of all parameters.

4. State evolution in coin subspace

To clear our main result further, we look in the following at the entanglement evolution of the wavefunction in the coin subspace. The complete wavefunction at time t consists of

two parts: one part in position space and additionally a part in coin subspace, $|\Psi(t)\rangle = \sum_x |x\rangle \otimes |\psi_{\text{coin}}(x)\rangle$. We will have a closer look at the states $|\psi_{\text{coin}}(x)\rangle$. With an artificial normalization of these states at each site one can calculate the global entanglement measure Q [13].

We note the following results from our simulations: (a) for all starting coin states the entanglement structure is constant in time at the starting point on the lattice; (b) at other lattice points, we observe entanglement oscillations. These two effects are shown in figure 4, with the γ GHZ state as initial state. If we sit on the starting point at $x = 60$, the entanglement as measured by the global entanglement Q is constant in time. If we go to $x = 50$ or $x = 40$ we can observe the described entanglement oscillations.

Another effect is shown in figure 5. The entanglement distribution over the lattice is plotted for $t = 50$ and the γ GHZ state as initial state for two different values of γ . For $\gamma = 1/\sqrt{2}$ the mean value in position space is 0 and the entanglement distribution over the lattice is symmetric. If we take $\gamma = 0.3$, the mean value is not equal to 0, and the entanglement distribution is asymmetric.

5. Conclusions and outlook

In conclusion, we have shown that for initial states with a special entanglement structure, the resulting mean value of a multiple coin quantum-walk scheme measures the entanglement of the initial coin state. We assume that our proposed connection is valid for what we call pure entangled states, states with only one kind of entanglement. Further it is shown that the symmetry of the probability distribution is reflected by the entanglement distribution.

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